

# BOUNDS FOR SMOOTH FANO WEIGHTED COMPLETE INTERSECTIONS

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**ABSTRACT.** We prove that if a smooth variety with non-positive canonical class can be embedded into a weighted projective space of dimension  $n$  as a well formed complete intersection and it is not an intersection with a linear cone therein, then the weights of the weighted projective space do not exceed  $n + 1$ . Based on this bound we classify all smooth Fano complete intersections of dimensions 4 and 5, and compute their invariants.

## 1. INTRODUCTION

Fano varieties are one of the important classes of algebraic varieties, both from birational and biregular points of view. It is known that smooth Fano varieties of a given dimension are bounded, see [KMM92, Theorem 0.2], so that one can hope for their explicit classification (actually this is known also for  $\varepsilon$ -log terminal Fano varieties, see [Bi16], but in this case any kind of explicit classification is hardly possible even in dimension 3). The only smooth Fano curve is  $\mathbb{P}^1$ . Smooth Fano varieties of dimension 2 are known as del Pezzo surfaces, and they were classified long ago. Smooth Fano threefolds were classified by V. Iskovskikh (see [Is77], [Is78], or [IP99, §12]), and S. Mori and S. Mukai (see, [MM82] and [Mu88]). The most important and hard part of this classification concerns Fano varieties with Picard rank 1. In dimension 3 such varieties (at least if they are general in the corresponding deformation family) appear to be either complete intersections in a weighted projective space, or zero loci of sections of homogeneous vector bundles on Grassmannians. In dimensions 4 and higher no complete classification is known, and at the moment no reasonable approach to the problem is yet in sight. Still there are partial classification results, including the list of all smooth Fano fourfolds of index at least 2 or Picard rank greater than one and Fano varieties of high coindex (see [Fu80], [Fu81], [Fu84], [Mu84], [Wi87], [Mu89], [Wi90]), smooth Fano fourfolds that are zero loci of sections of homogeneous vector bundles on Grassmannians (see [Kü95], [Kü97, §4], [Ku16]), smooth Fano fourfolds that are weighted complete intersections (see [Kü97, Proposition 2.2.1]), and some other sporadic results (see [Kü97, §3]). The purpose of this paper is to study smooth Fano weighted complete intersections, and give effective numerical bounds that allow to classify them.

To be able to classify weighted complete intersections of a given dimension satisfying some nice properties, one needs an effective bound on the corresponding discrete parameters. In [CCC11, Theorem 1.3] such bound was obtained for codimension of a quasi-smooth (see Definition 2.4 below) weighted complete intersection. In [Ch15, Theorem 1.3] the degrees were bounded in terms of canonical volume and discrepancies.

We will be interested in the case of smooth Fano varieties that can be described as complete intersections in weighted projective spaces. Also, we will deal with the case

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The authors are supported by RSF grant, project 14-21-00053 dated 11.08.14.

when our weighted complete intersection is Fano or Calabi–Yau. In both cases by adjunction formula it is actually enough to bound the weights of the corresponding weighted projective space.

Let  $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$  be a weighted projective space, and  $X \subset \mathbb{P}$  be a weighted complete intersection of multidegree  $(d_1, \dots, d_c)$  for some  $c \geq 0$ . We will usually assume that  $X$  is *not an intersection with a linear cone*, i.e. one has  $d_j \neq a_i$  for all  $i$  and  $j$ , cf. Remark 2.8. Finally, it is convenient to assume that  $X$  is *well formed*, see Definition 2.3 and Theorem 2.9 below.

The main result of this paper is the following.

**Theorem 1.1.** *Let  $X \subset \mathbb{P}(a_0, \dots, a_n)$ ,  $n \geq 2$ , be a smooth well formed weighted complete intersection of multidegree  $(d_1, \dots, d_c)$ . Suppose that  $X$  is not an intersection with a linear cone. If  $X$  is Fano, then for every  $0 \leq i \leq n$  one has  $a_i \leq n$ , and for every  $1 \leq j \leq c$  one has  $d_j \leq n(n+1)$ . Similarly, if  $X$  is Calabi–Yau, then for every  $0 \leq i \leq n$  one has  $a_i \leq n+1$ , and for every  $1 \leq j \leq c$  one has  $d_j \leq (n+1)^2$ .*

To prove Theorem 1.1 we use the following approach. Exploiting smoothness assumption, we write down a bunch of necessary conditions on the parameters  $a_i$  and  $d_j$ , which appear to be inequalities (sometimes involving products of weights or degrees). On the other hand, Fano or Calabi–Yau condition implies an inequality between the sums of  $a_i$  and  $d_j$ . Then we treat all these inequalities as if  $a_i$  and  $d_j$  were arbitrary real numbers, and solve the corresponding optimization problem using the standard down-to-earth method of Lagrange multipliers.

The bounds given by Theorem 1.1 are sharp for an infinite set of dimensions, see Remark 3.2 below.

Using Theorem 1.1, we will give a classification of weighted Fano complete intersections of dimensions 4 and 5, see §5 below. Note that a classification of four-dimensional weighted Fano complete intersections was already obtained by O. Küchle in [Kü97, Proposition 2.2.1]; his method builds on a classification of weighted homogeneous polarized Calabi–Yau complete intersections, see [Og91, Main Theorem II]. In a way this is closer to the methods that were classically used in a classification of Fano threefolds, but one can hardly expect that it can be easily generalized to higher dimensions.

The plan of the paper is as follows. In §2 we recall some basic properties of weighted complete intersections. In §3 we prove Theorem 1.1. In §4 we explain the (well known) method that can be used to compute Hodge numbers of smooth weighted complete intersections. In §5 we provide a classification of smooth Fano weighted complete intersections of dimensions 4 and 5. Finally, in Appendix A we collect some (nearly elementary) auxiliary material used in §3.

**Notation and conventions.** All varieties are compact and are defined over the field of complex numbers  $\mathbb{C}$ . For a bigraded ring  $R$  we denote its  $(p, q)$ -component by  $R_{(p,q)}$ . For a weighted complete intersection  $X$  in  $\mathbb{P}(a_0, \dots, a_n)$  of multidegree  $(d_1, \dots, d_c)$  the number  $\sum a_i - \sum d_j$  is denoted by  $I(X)$ .

**Acknowledgments.** We are grateful to K. Besov, A. Corti, A. Kuznetsov, T. Okada, and Yu. Prokhorov for useful discussions, and to A. Harder who helped us to compute Hodge numbers in §5.

## 2. SMOOTHNESS

We recall here some basic properties of weighted complete intersections. We refer the reader to [Do82] and [IF00] for more details. Put

$$\mathbb{P} = \mathbb{P}(a_0, \dots, a_n) = \text{Proj} \mathbb{C}[x_0, \dots, x_n],$$

where the weight of  $x_i$  equals  $a_i$ .

**Definition 2.1** (see [IF00, Definition 5.11]). The weighted projective space  $\mathbb{P}$  is said to be *well formed* if the greatest common divisor of any  $n$  of the weights  $a_i$  is 1.

Any weighted projective space is isomorphic to a well formed one, see [Do82, 1.3.1].

**Lemma 2.2** (see [IF00, 5.15]). *The singular locus of  $\mathbb{P}$  is a union of strata*

$$\Lambda_J = \{(x_0 : \dots : x_n) \mid x_j = 0 \text{ for all } j \notin J\}$$

*for all subsets  $J \subset \{0, \dots, n\}$  such that the greatest common divisor of the weights  $a_j$  for  $j \in J$  is greater than 1.*

**Definition 2.3** (see [IF00, Definition 6.9]). A subvariety  $X \subset \mathbb{P}$  of codimension  $c$  is said to be *well formed* if  $\mathbb{P}$  is well formed and

$$\text{codim}_X (X \cap \text{Sing } \mathbb{P}) \geq 2.$$

The following notion is a replacement of smoothness suitable for subvarieties of weighted projective spaces.

**Definition 2.4** (see [IF00, Definition 6.3]). Let  $p: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}$  be the natural projection. A subvariety  $X \subset \mathbb{P}$  is said to be *quasi-smooth* if  $p^{-1}(X)$  is smooth.

We say that a variety  $X \subset \mathbb{P}$  of codimension  $c$  is a *weighted complete intersection of multidegree*  $(d_1, \dots, d_c)$  if its weighted homogeneous ideal in  $\mathbb{C}[x_0, \dots, x_n]$  is generated by a regular sequence of  $c$  homogeneous elements of degrees  $d_1, \dots, d_c$ . Note that in general a weighted complete intersection is not even locally a complete intersection in the usual sense.

*Remark 2.5.* It is possible that a weighted complete intersection of a given multidegree in  $\mathbb{P}$  does not exist, even if  $c$  is small. For example, there is no such thing as a hypersurface of degree  $d < \min(a_0, \dots, a_n)$  in  $\mathbb{P}$ , or a weighted complete intersection of multidegree  $(2, 2)$  in  $\mathbb{P}(1, 3, 4, 5)$ .

Singularities of quasi-smooth well formed weighted complete intersections can be easily described.

**Proposition 2.6** (see [Di86, Proposition 8]). *Let  $X \subset \mathbb{P}$  be a quasi-smooth well formed weighted complete intersection. Then the singular locus of  $X$  is the intersection of  $X$  with the singular locus of  $\mathbb{P}$ .*

*Remark 2.7.* Note that the definition of “general position” in [Di86] coincides with our definition of well formedness.

Recall that the weighted complete intersection  $X$  is said to be an intersection with a linear cone if one has  $d_j = a_i$  for some  $i$  and  $j$ .

*Remark 2.8.* If this condition fails, one can exclude the  $i$ -th weighted homogeneous coordinate and think about  $X$  as a weighted complete intersection in a weighted projective space of lower dimension, provided that  $X$  is general enough, cf. Remark 5.2 below. Note however that in general this new weighted projective space may fail to be well formed, and the new weighted complete intersection may fail to be nice in other ways as well.

It appears that the assumptions that the complete intersection is well formed, quasi-smooth, and is not an intersection with a linear cone are not always independent. In principle it can allow us to drop some of the assumptions in the rest of the paper, but we will refrain from doing so to keep the assertions more explicit.

**Theorem 2.9** (see [IF00, Theorem 6.17]). *Suppose that the weighted projective space  $\mathbb{P}$  is well formed. Then any quasi-smooth complete intersection of dimension at least 3 in  $\mathbb{P}$  is either an intersection with a linear cone or well formed.*

There is the following version of the adjunction formula that holds for quasi-smooth well formed weighted complete intersections.

**Lemma 2.10** (see [Do82, Theorem 3.3.4], [IF00, 6.14]). *Let  $X \subset \mathbb{P}$  be a quasi-smooth well formed weighted complete intersection of multidegree  $(d_1, \dots, d_c)$ . Then*

$$\omega_X = \mathcal{O}_X \left( \sum d_i - \sum a_j \right).$$

Along with quasi-smooth weighted complete intersections one may consider those weighted complete intersections that are smooth in the usual sense.

**Proposition 2.11.** *Let  $X \subset \mathbb{P}$  be a smooth well formed weighted complete intersection. Then  $X$  does not pass through singular points of  $\mathbb{P}$ .*

*Proof.* Suppose that  $X$  contains a singular point  $P$  of  $\mathbb{P}$ . Let  $U \subset \mathbb{P}$  be an affine neighborhood of  $P$ , and  $\pi: \tilde{U} \rightarrow U$  be its natural finite cover (see [IF00, 5.3]), so that  $\tilde{U}$  is isomorphic to an open subset of  $\mathbb{A}^n$ , and  $\pi$  is a quotient by a group  $\mathbb{Z}/r\mathbb{Z}$  for some  $r > 1$ . Put  $V = X \cap U$ . Let  $\Sigma$  be the singular locus of  $U$ . Since  $X$  is well formed, the intersection of  $\Sigma$  with  $V$  has codimension at least 2 in  $V$ . Let  $\tilde{V}$  be the preimage of  $V$  with respect to  $\pi$ , and let  $\pi_V: \tilde{V} \rightarrow V$  be the corresponding finite cover. Then  $\tilde{V}$  is a complete intersection in  $U$ . Note also that  $\pi_V$  is étale outside of  $\Sigma$ , and thus  $\tilde{V}$  is smooth in codimension 1. In particular,  $\tilde{V}$  is Cohen–Macaulay, see [Ei95, §18.5].

Put  $V^\circ = V \setminus \Sigma$ , and let  $\tilde{V}^\circ$  be the preimage of  $V^\circ$  with respect to  $\pi_V$ . Since  $V$  is smooth and  $\dim V > 1$ , there exists a simply connected punctured neighborhood of  $P$  in  $V$ , and hence there also exists a simply connected punctured neighborhood of  $P$  in  $V^\circ$ . Thus the morphism  $\pi_V$  is étale in some punctured neighborhood  $\mathcal{V}$  of the point  $\pi^{-1}(P)$  in  $\tilde{V}^\circ$  whose image with respect to  $\pi$  is simply connected. The complex space  $\tilde{V}$  splits into a union of its connected components  $\tilde{V}_1, \dots, \tilde{V}_r$ . Thus in a neighborhood of the point  $\pi^{-1}(P)$  the variety  $\tilde{V}$  splits into a union of its irreducible components  $\tilde{V}_1, \dots, \tilde{V}_r$ . The irreducible component  $\tilde{V}_1$  intersects all other irreducible components (in particular) at the point  $\pi^{-1}(P)$ , so that  $\tilde{V}$  is connected. Since  $\tilde{V}$  is Cohen–Macaulay, by [Ei95, Theorem 18.12] there is an irreducible component  $\tilde{V}_k$  such that the intersection  $Z = \tilde{V}_1 \cap \tilde{V}_k$  has codimension 1 in  $\tilde{V}_1$ . The variety  $\tilde{V}$  is singular along  $Z$ . Since  $\pi_V$  is étale at a general point of  $Z$ , we conclude that  $V$  is singular at a general point of  $\pi_V(Z)$ , which is a contradiction.  $\square$

*Remark 2.12.* A. Kuznetsov pointed out that there is an alternative proof of Proposition 2.11 that is purely algebraic and does not depend on the base field. Namely, in the notation of our proof of Proposition 2.11 the variety  $\tilde{V}$  is normal (since it is a locally complete intersection smooth in codimension 1). Since  $V$  is smooth by assumption, the branch locus  $R$  of  $\pi_V$  has codimension 1 in  $V$  by the purity of the branch locus, see [Au62, Theorem 1.4]. Now we can obtain a contradiction as above. Still we prefer to keep the original proof of Proposition 2.11, since we believe that it makes the geometric reason explaining why this property holds more evident, and our base field is  $\mathbb{C}$  anyway.

*Remark 2.13.* The assertion of Proposition 2.11 fails without the assumption that  $X$  is well formed. Indeed, suppose that  $a_0 = a_1 = 1$  and  $a_2 = \dots = a_n = 2$ ; put  $c = 1$  and  $d_1 = 2$ . Then  $X$  is not well formed, but it is smooth since it is isomorphic to  $\mathbb{P}^{n-1}$ . However, it passes through singular points of  $\mathbb{P}$ . For  $n = 2$  this example gives a line on a usual quadratic cone.

As an application of Proposition 2.11 one can show that being smooth is a stronger condition than being quasi-smooth, provided that we work with well formed weighted complete intersections.

**Corollary 2.14.** *Let  $X \subset \mathbb{P}$  be a smooth well formed weighted complete intersection. Then  $X$  is quasi-smooth.*

*Proof.* The morphism  $p: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}$  is a locally trivial  $\mathbb{C}^*$ -bundle over the non-singular part  $U$  of  $\mathbb{P}$ , while  $X$  is contained in  $U$  by Proposition 2.11. Hence  $p$  is a locally trivial  $\mathbb{C}^*$ -bundle over  $X$ , and thus the preimage of  $X$  with respect to  $p$  is smooth, which means that  $X$  itself is quasi-smooth.  $\square$

Another consequence of Proposition 2.11 is the following result.

**Lemma 2.15** (cf. [IF00, 6.12]). *Let  $X \subset \mathbb{P}$  be a smooth well formed weighted complete intersection of multidegree  $(d_1, \dots, d_c)$ . Then for every  $k$  and every choice of  $k$  weights  $a_{i_1}, \dots, a_{i_k}$ ,  $i_1 < \dots < i_k$ , such that their greatest common divisor  $\delta$  is greater than 1 there exist  $k$  degrees  $d_{s_1}, \dots, d_{s_k}$ ,  $s_1 < \dots < s_k$ , such that their greatest common divisor is divisible by  $\delta$ .*

*Proof.* Choose a positive integer  $k$ , and suppose that there are  $k$  weights  $a_{i_1}, \dots, a_{i_k}$  with  $i_1 < \dots < i_k$ , such that their greatest common divisor  $\delta$  is greater than 1. Let  $t$  be the number of degrees  $d_j$  that are divisible by  $\delta$ . Suppose that  $t < k$ . We claim that in this case  $X$  is singular. Indeed, let  $J$  be the set of indices  $j$  such that  $d_j$  is divisible by  $\delta$ , and let  $\Lambda$  be the subvariety in  $\mathbb{P}$  given by equations  $x_j = 0$  for  $j \in J$ . Then for any  $j' \notin J$  the polynomial  $F_{j'}$  does not contain monomials that depend only on  $x_j$  with  $j \in J$ . On the other hand, the equations  $F_j = 0$  for  $j \in J$  cut out a non-empty subset of  $\Lambda$ . Since  $\delta > 1$ , the weighted projective space  $\mathbb{P}$  is singular along  $\Lambda$ , see Lemma 2.2. This gives a contradiction with Proposition 2.11.  $\square$

*Remark 2.16.* The condition provided by Lemma 2.15 is only necessary for the weighted complete intersection  $X$  to be smooth, but not sufficient. For example, assume that  $a_0 = \dots = a_r = 1$ , while  $2 < a_{r+1} \leq \dots \leq a_n$ , and  $a_j$  are pairwise coprime. Choose  $d_1$  and  $d_2$  so that  $d_2$  is divisible by all  $a_j$ , and  $2 \leq d_1 < a_{r+1}$ . Then a general weighted complete intersection  $X$  of multidegree  $(d_1, d_2)$  in  $\mathbb{P}$  is not smooth provided that  $r \leq n - 2$ ; moreover, it is reducible if  $r = 1$ , and non-reduced if  $r = 0$ . Another way how smoothness

may fail is illustrated by an example of a weighted complete intersection  $X$  of multidegree  $(2, 30)$  in  $\mathbb{P}$  when  $a_0 = \dots = a_{n-2} = 1$ ,  $a_{n-1} = 6$ , and  $a_n = 10$ ; in this case we see that the assertion of Proposition 2.11 does not hold, so that  $X$  is singular. See also Lemma 3.1(i) below.

### 3. WEIGHT BOUND

In this section we derive Theorem 1.1 from elementary results of Appendix A. The method we use here is somewhat similar to [CS13, §3].

Let  $X \subset \mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ ,  $n \geq 2$ , be a smooth well formed weighted complete intersection of multidegree  $(d_1, \dots, d_c)$  that is not an intersection with a linear cone. We can assume that  $X$  is *normalized*, i.e. that inequalities  $a_0 \leq \dots \leq a_n$  and  $d_1 \leq \dots \leq d_c$  hold. Moreover, if  $c = 0$ , then one has  $X \cong \mathbb{P} \cong \mathbb{P}^n$ , and there is nothing to prove; therefore, we will always assume that  $c \geq 1$ .

We need an auxiliary result that is easy to establish and well known to experts.

**Lemma 3.1.** *Let  $X \subset \mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ ,  $n \geq 2$ , be a smooth well formed normalized weighted complete intersection of multidegree  $(d_1, \dots, d_c)$  that is not an intersection with a linear cone. Then the following assertions hold.*

- (i) *One has  $d_{c-k} > a_{n-k}$  for all  $1 \leq k \leq c-1$ .*
- (ii) *One has  $d_c \geq 2a_n$ .*
- (iii) *The integer  $a_0 \cdot \dots \cdot a_n$  divides the integer  $d_1 \cdot \dots \cdot d_c$ .*

*Proof.* Assertion (i) is given by [IF00, Lemma 18.14] and holds under a weaker assumption of quasi-smoothness. If  $a_n = 1$ , then the remaining assertions of the lemma obviously hold, and thus we can assume that  $a_n > 1$ .

Let  $x_0, \dots, x_n$  be homogeneous coordinates on  $\mathbb{P}$  of weights  $a_0, \dots, a_n$ , respectively, and let  $f_1 = \dots = f_c = 0$  be equations of  $X$  in  $\mathbb{P}$ .

Suppose that  $d_c < 2a_n$ . Then none of  $f_j$  contains a monomial  $x_n^r$  with non-zero coefficient if  $r \geq 2$ . Also, since  $X$  is not an intersection with a linear cone, none of  $f_j$  contains a monomial  $x_n$  with non-zero coefficient either. Therefore, we see that every  $f_j$  vanishes at the point  $P$  given by  $x_0 = \dots = x_{n-1} = 0$ , so that  $X$  passes through  $P$ . On the other hand,  $P$  is a singular point of  $\mathbb{P}$  by Lemma 2.2 because  $a_n > 1$ . Thus Proposition 2.6 implies that  $X$  is singular at  $P$ , which is a contradiction. This gives assertion (ii).

To prove assertion (iii), choose a prime number  $p$ , and denote by  $\nu_p^{(r)}(a_0, \dots, a_n)$  the number of the weights  $a_i$  that are divisible by  $p^r$ . Similarly, denote by  $\nu_p^{(r)}(d_1, \dots, d_c)$  the number of the degrees  $d_j$  that are divisible by  $p^r$ . By Lemma 2.15 for every  $r$  one has

$$\nu_p^{(r)}(a_0, \dots, a_n) \leq \nu_p^{(r)}(d_1, \dots, d_c).$$

This implies that the  $p$ -adic valuation of the integer  $a_0 \cdot \dots \cdot a_n$  does not exceed the  $p$ -adic valuation of the integer  $d_1 \cdot \dots \cdot d_c$ . Since this holds for an arbitrary prime  $p$ , we obtain assertion (iii).  $\square$

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Put  $N = n+1$ . Denote  $A_{i+1} = a_{n-i}$  for  $0 \leq i \leq n$ , and  $D_j = d_{c-j+1}$  for  $1 \leq j \leq c$ . Then one has  $A_1 \geq \dots \geq A_N$  and  $D_1 \geq \dots \geq D_c$ . Moreover, by

Lemma 3.1(i) one has  $D_2 > A_2, \dots, D_c > A_c$ . By Lemma 3.1(ii) we also have  $D_1 \geq 2A_1$ , and by Lemma 3.1(iii) we have

$$A_1 \cdot \dots \cdot A_N \leq D_1 \cdot \dots \cdot D_c.$$

Put

$$L = a_0 + \dots + a_n - d_1 - \dots - d_c = A_1 + \dots + A_N - D_1 - \dots - D_c.$$

Then  $L > 0$  provided that  $X$  is Fano, and  $L \geq 0$  provided that  $X$  is Calabi–Yau. This follows from Lemma 2.10.

Suppose that  $X$  is a Fano variety. Then  $N \geq 2c + 1$  by [CCC11, Theorem 1.3]. Therefore, Proposition A.11 implies that  $A_1 \leq N - 1$ , which can be rewritten as  $a_n \leq n$ .

Now suppose that  $X$  is Calabi–Yau. Then a general weighted complete intersection of multidegree  $d_1, \dots, d_c$  in  $\mathbb{P}(1, a_0, \dots, a_n)$  is a smooth well formed Fano weighted complete intersection that is not an intersection with a linear cone. Thus one has  $a_n \leq n + 1$ .

Since  $X$  is normalized, we obtain similar inequalities for all other weights  $a_i$ . Finally, the inequalities for the degrees  $d_j$  follow from the fact that  $L$  is positive if  $X$  is Fano, and non-negative if  $X$  is Calabi–Yau. This completes the proof of Theorem 1.1.  $\square$

*Remark 3.2.* The bounds for  $a_n$  given by Theorem 1.1 are sharp for an infinite set of dimensions. Indeed, if  $n$  is odd,  $a_0 = \dots = a_{n-2} = 1$ ,  $a_{n-1} = 2$ , and  $a_n = n$ , then a general hypersurface of weighted degree  $2n$  in  $\mathbb{P}$  is a smooth well formed Fano weighted complete intersection. Similarly, if  $n$  is even,  $a_0 = \dots = a_{n-2} = 1$ ,  $a_{n-1} = 2$ , and  $a_n = n + 1$ , then a general hypersurface of weighted degree  $2n + 2$  in  $\mathbb{P}$  is a smooth well formed Calabi–Yau weighted complete intersection.

Although Remark 3.2 shows that the bound for the maximal weight  $a_n$  given by Theorem 1.1 is more or less sharp, there are stronger bounds for some other weights  $a_i$  in certain cases.

**Lemma 3.3.** *Let  $X \subset \mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ ,  $n \geq 2$ , be a smooth well formed normalized weighted complete intersection of multidegree  $(d_1, \dots, d_c)$  that is not an intersection with a linear cone. Suppose that  $X$  is Fano. Then for every  $1 \leq k \leq \dim X$  one has*

$$a_k \leq 2^{\frac{\dim X}{\dim X - k + 1}}.$$

Moreover, one has  $a_0 = a_1 = 1$ .

*Proof.* By [CCC11, (2.6)] and Lemma 3.1(iii) one has

$$\frac{\left(\frac{c + \dim(X) + 1 - I(X)}{c}\right)^c}{\prod_{i=0}^{\dim(X)} a_i} \geq \frac{\prod_{j=1}^c d_j}{\prod_{k=0}^n a_k} \geq 1.$$

Since  $X$  is Fano, one has  $I(X) > 0$ , so that

$$\left(1 + \frac{\dim(X)}{c}\right)^c = \left(\left(1 + \frac{\dim(X)}{c}\right)^{\frac{c}{\dim(X)}}\right)^{\dim(X)} \geq \prod_{i=0}^{\dim(X)} a_i.$$

Suppose that for some  $1 \leq k \leq \dim X$  the inequality

$$a_k > 2^{\frac{\dim X}{\dim X - k + 1}}$$

holds. Then

$$\prod_{i=0}^{\dim(X)} a_i \geq a_k^{\dim X - k + 1} > 2^{\dim(X)},$$

and thus

$$\left(1 + \frac{\dim(X)}{c}\right)^{\frac{c}{\dim(X)}} > 2.$$

The latter means that  $\frac{c}{\dim(X)} > 1$  and gives a contradiction with [CCC11, Theorem 1.3].

Now suppose that  $a_1 > 1$ . To avoid a contradiction with [CCC11, Theorem 1.3] one must have  $a_0 = 1$  and  $a_1 = \dots = a_{\dim(X)} = 2$ , and the inequalities in the above argument must become equalities. In particular, one has  $\dim(X) = c$ . Moreover, Lemma 2.15 implies that all  $d_j$  are divisible by 2, and  $a_i$  for  $i > \dim(X)$  are not. Now Lemma 3.1(i),(ii) implies that for all  $0 \leq k \leq c - 1$  one has  $d_{c-k} \geq 2a_{n-k}$ , so that

$$\sum_{j=1}^c d_j \geq \sum_{i=\dim(X)+1}^n 2a_i \geq \left( \sum_{i=\dim(X)+1}^n a_i \right) + 2c + 1 = \sum_{i=0}^n a_i.$$

This contradicts the assumption that  $X$  is Fano.  $\square$

#### 4. HODGE NUMBERS

The idea of description of Hodge numbers for complete intersections in weighted projective spaces as dimensions of graded components of particular (bi)graded rings goes back to [Gr69], [St77], [Do82], [PSt83]; another approach, due to Hirzebruch, can be found in [SGA7, Exp. XI, Theorem 2.2]. For complete intersections in toric varieties one can look at [BC94]. The way of the computation called the *Cayley trick* is to relate the Hodge structure of a complete intersection to the Hodge structure of some higher-dimensional hypersurface. We describe this approach following [Ma99].

Let  $Y$  be a simplicial toric variety of dimension  $n$ . Let  $D_1, \dots, D_b$  be its prime boundary divisors. Denote the group of  $r$ -cycles on  $Y$  modulo rational equivalence by  $A_r(Y)$ . Consider an  $A_{n-1}(Y)$ -graded ring  $R_0 = \mathbb{C}[x_1, \dots, x_b]$  with grading defined by

$$\deg_A \left( \prod_{i=1}^b x_i^{r_i} \right) = \sum_{i=1}^b r_i D_i.$$

One has  $\text{Spec}(R_0) \cong \mathbb{A}^b$ , and there is a natural correspondence between rays  $e_i$  of a fan of  $Y$  and variables  $x_i$ . Define a subvariety  $Z$  in  $\text{Spec}(R_0)$  as a union of hypersurfaces  $\{\prod x_i = 0 \mid e_i \notin \sigma\}$  over all cones  $\sigma$  of a fan of  $Y$ . Then  $Y$  is a geometric quotient of  $\text{Spec}(R_0) \setminus Z \subset \mathbb{A}^b$  by the torus  $\mathbb{D} = \text{Hom}_{\mathbb{Z}}(A_{n-1}(Y), \mathbb{C}^*)$ . We call a polynomial  $f \in R_0$  homogeneous if all its monomials are of degree  $d$  for some  $d \in A_{n-1}(Y)$ . For any homogeneous polynomials  $f_1, \dots, f_c$  their common zero set intersected with  $U$  is stable under the action of  $\mathbb{D}$  so they determine a closed subset  $X$  in  $Y$ .

Consider a ring  $R = \mathbb{C}[x_1, \dots, x_b, y_1, \dots, y_c]$ . Choose  $c$  homogeneous polynomials  $f_1, \dots, f_c \in R_0 \subset R$  with  $\deg_A(f_i) = d_i \in A_{n-1}(Y)$ . Define a bigrading on  $R$  with values in  $A_{n-1}(Y) \times \mathbb{Z}$  by  $\text{bideg}(x_i) = (D_i, 0)$  and  $\text{bideg}(y_i) = (-d_i, 1)$ . Consider the polynomial  $F = y_1 f_1 + \dots + y_c f_c$ . Obviously, one has  $\text{bideg}(F) = (0, 1)$ . Define a Jacobian



ideal

$$J = \left( \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_b}, \frac{\partial F}{\partial y_1}, \dots, \frac{\partial F}{\partial y_c} \right)$$

and a bigraded ring  $R(F) = R/J$ .

We will assume that the subvariety  $X$  defined by the polynomials  $f_1, \dots, f_c$  is *quasi-smooth*. Recall from [Ma99, Definition 1.1] that this means that a common zero set of  $f_1, \dots, f_c$  inside  $U$  is a smooth subvariety of codimension  $c$ , cf. Definition 2.4. In this case  $X$  has a pure Hodge structure on its cohomology, see [Ma99, §3]. In particular, one can speak about Hodge numbers  $h^{p,q}(X)$ .

Define  $c_k$  as a number of cones of dimension  $k$  in the fan of  $Y$ . Put

$$l_k = \sum_{i=k}^n (-1)^{i-k} \binom{i}{k} c_{n-i}.$$

**Theorem 4.1** (see [Da78, Theorem 10.8 and Remark 10.9]). *Let  $Y$  be a simplicial toric variety of dimension  $n$ . Then  $\dim H^{2k}(Y, \mathbb{C}) = l_k$  for all  $0 \leq k \leq n$ .*

There is an analog of Lefschetz hyperplane section theorem for complete intersections in simplicial toric varieties, see e.g. [Ma99, Proposition 1.4]. In particular, the only Hodge numbers of such complete intersection  $X$  that are not inherited from the ambient toric variety are  $h^{p,q}(X)$  with  $p + q = \dim(X)$ .

**Theorem 4.2** (see [Ma99, Theorem 3.6]). *Let  $Y$  be a simplicial toric variety of dimension  $n$  and let  $D_1, \dots, D_b$  be its boundary divisors. Let  $X \subset Y$  be a quasi-smooth complete intersection of ample divisors defined by homogeneous polynomials  $f_1, \dots, f_c$  with  $\deg_A(f_i) = d_i \in A_{n-1}$ . Suppose that  $\dim X = n - c \geq 3$ . Denote*

$$\sum_{r=1}^b D_r - \sum_{s=1}^c d_s \in A_{n-1}(X)$$

by  $i(X)$ . Then for  $p \neq \frac{n-c+1}{2}$  and  $p \neq \frac{n-c}{2}$  one has

$$h^{n-c-p,p}(X) = \dim R(F)_{(-i(X),p)}.$$

For  $p = \frac{n-c+1}{2}$  one has

$$h^{p-1,p}(X) = \dim R(F)_{(-i(X),p)} + l_{p-c} - l_p.$$

For  $p = \frac{n-c}{2}$  one has

$$h^{p,p}(X) = \dim R(F)_{(-i(X),p)} + l_p.$$

In a particular case of complete intersections in a weighted projective space the even cohomology spaces  $H^{2k}(Y, \mathbb{C})$  are one-dimensional, see [Do82, Corollary 2.3.6]. This allows to simplify Theorem 4.2 in this case. Recall that for a weighted complete intersection  $X$  in  $\mathbb{P}(a_0, \dots, a_n)$  of multidegree  $(d_1, \dots, d_c)$  we denote the number  $\sum a_i - \sum d_j$  by  $I(X)$ .

**Corollary 4.3.** *Let  $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$  be a well formed weighted projective space, and  $X \subset \mathbb{P}$  be a quasi-smooth weighted complete intersection defined by homogeneous polynomials  $f_1, \dots, f_c$  with  $\deg(f_i) = d_i$ . Suppose that  $\dim X = n - c \geq 3$ . Then for  $p \neq \frac{n-c}{2}$  one has*

$$h^{n-c-p,p}(X) = \dim R(F)_{(-I(X),p)}$$

and for  $p = \frac{n-c}{2}$  one has

$$h^{p,p}(X) = \dim R(F)_{(-I(X),p)} + 1.$$

**Example 4.4.** Consider the weighted projective space  $\mathbb{P} = \mathbb{P}(1, 1, 1, 1, 1, 1, 3)$  with weighted homogeneous coordinates  $x_0, \dots, x_6$ , where the weights of  $x_0, \dots, x_5$  equal 1, and the weight of  $x_6$  equals 3. Let  $X$  be a (general) weighted complete intersection of hypersurfaces of degrees 2 and 6 in  $\mathbb{P}$  given by polynomials  $f_1$  and  $f_2$ , respectively. Thus  $F = y_1 f_1 + y_2 f_2$  and

$$J = \left( \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_6}, f_1, f_2 \right).$$

One has

$$\begin{aligned} \text{bideg}(x_0) = \dots = \text{bideg}(x_5) &= (1, 0), \text{bideg}(x_6) = (3, 0), \\ \text{bideg}(y_1) &= (-2, 1), \text{bideg}(y_2) = (-6, 1). \end{aligned}$$

Since  $I(X) = 1$ , one gets  $h^{1,3}(X) = \dim R(F)_{(-1,1)}$ . The component  $R(F)_{(-1,1)}$  is generated by polynomials of type  $g_1 y_1 + g_5 y_2$ , where  $g_s$  are polynomials of degrees  $s$  in  $x_i$ . There are 6 parameters for  $g_1$  and  $\binom{10}{5} + \binom{7}{5} = 273$  parameters for  $g_5$ , so  $R_{(-1,1)} = 279$ . There are no polynomials from  $R_{(-1,1)}$  that are divisible by  $f_2$ , and  $\binom{8}{5} = 56$  parameters for polynomials in  $x_0, \dots, x_5, y_2$  that are divisible by  $f_1$ . Up to scaling there is a unique polynomial that is divisible by  $f_1$  and  $x_6$ . Moreover, one has  $\text{bideg}(\frac{\partial F}{\partial x_i}) = (-1, 1)$  for  $i = 1, \dots, 6$ , so there are 6 parameters for polynomials from  $R_{(-1,1)}$  that are divisible by  $\frac{\partial F}{\partial x_i}$ . Similarly, one has  $\text{bideg}(\frac{\partial F}{\partial x_6}) = (-3, 1)$ , so there are  $\binom{7}{5} = 21$  parameters for polynomials from  $R_{(-1,1)}$  that are divisible by  $\frac{\partial F}{\partial x_6}$ . One of them, namely  $\frac{\partial F}{\partial x_6} f_1$ , is already taken into account. Thus

$$h^{1,3}(X) = \dim R(F)_{(-1,1)} = \dim R_{(-1,1)} - 56 - 1 - 6 - 21 + 1 = 279 - 56 - 1 - 6 - 21 + 1 = 196.$$

One can obtain the following elegant formula for  $\dim H^0(X, \mathcal{O}_X(k))$ .

**Theorem 4.5** ([Do82, Theorem 3.4.4]). *Let  $X$  be a quasi-smooth well formed weighted complete intersection of multidegree  $(d_1, \dots, d_c)$  in  $\mathbb{P}(a_0, \dots, a_n)$ . Then*

$$\sum_{k=0}^{\infty} (\dim H^0(X, \mathcal{O}_X(k))) t^k = \frac{\prod_{s=0}^c (1 - t^{d_s})}{\prod_{r=0}^n (1 - t^{a_r})}.$$

For a weighted projective space  $\mathbb{P}(a_0, \dots, a_n)$  with weighted homogeneous coordinates  $x_0, \dots, x_n$  denote by  $P(r)$  the dimension of the vector space of (weighted) homogeneous polynomials in  $x_0, \dots, x_n$  of (weighted) degree  $r$ . Applying Theorem 4.5 together with Lemma 2.10 one gets the following.

**Corollary 4.6.** *Let  $X$  be a quasi-smooth well formed weighted complete intersection of multidegree  $(d_1, \dots, d_c)$  in  $\mathbb{P}(a_0, \dots, a_n)$ . Then*

$$\dim H^0(X, -K_X) = \sum_{s=0}^c (-1)^s \sum_{1 \leq k_1 < \dots < k_s \leq c} P(I(X) - d_{k_1} - \dots - d_{k_s}).$$

Another approach to compute Hodge numbers for complete intersections in (usual) projective spaces is due to F. Hirzebruch. Define

$$H(d) = \frac{(y+1)^{d-1} - (z+1)^{d-1}}{(z+1)^d y - (y+1)^d z} = \frac{d-1 + \binom{d-1}{2}(y+z) + \binom{d-1}{3}(y^2 + yz + z^2) + \dots}{1 - \binom{d}{2}yz - \binom{d}{3}yz(y+z) + \dots}$$

and

$$H(d_1, \dots, d_c) = \sum_{Q \subset [1, c], Q \neq \emptyset} ((y+1)(z+1))^{|Q|-1} \prod_{i \in Q} H(d_i),$$

where  $|Q|$  is a number of elements of  $Q$ . If  $F$  is a formal series in two variables  $y$  and  $z$ , we denote by  $F^{(m)}$  the sum of monomials in  $F$  of homogeneous degree  $m$ .

**Theorem 4.7** (see [SGA7, Exp. XI, Théorème 2.3]). *Let  $X$  be a smooth complete intersection of hypersurfaces of degrees  $d_1, \dots, d_c$  in  $\mathbb{P}^n$ . Put  $m = \dim X = n - c$ . Then*

$$\sum h^{p, m-p}(X) y^p z^{m-p} = \left( H(d_1, \dots, d_c) + \delta y^{\frac{m}{2}} z^{\frac{m}{2}} \right)^{(m)},$$

where  $\delta = 1$  if  $m$  is even and  $\delta = 0$  if  $m$  is odd.

*Remark 4.8.* There is a conjectural approach to description of Hodge numbers of Fano varieties via their Landau–Ginzburg models, see [KKP14]. It was verified for del Pezzo surfaces, see [LP16]; for smooth toric varieties see [Ha16]. Its reformulation in terms of toric Landau–Ginzburg models for one of the Hodge numbers was checked for Picard rank one Fano threefolds (see [Prz13]) and partially for complete intersections in projective spaces (see [PS15]).

## 5. DIMENSIONS 4 AND 5

In this section we provide a classification of smooth well formed Fano weighted complete intersections of dimensions 4 and 5. This is done by a straightforward check using the bounds given by Theorem 1.1 and Lemma 3.3. To simplify conventions, we exclude the projective space (which is a codimension 0 smooth Fano complete intersection in itself) from our lists. Note that at this step we obtain only necessary conditions on the weights and degrees, and in each case one has to check that there actually exists a weighted complete intersection with the corresponding parameters, and that it is smooth. The Hodge numbers are computed using Corollary 4.3, and sometimes Theorem 4.7 when the latter is more convenient to apply. Since the Hodge numbers are constant in smooth families, for these computations it is enough to pick up one example in each of the families. In principle, all this can be done in an arbitrary given dimension.

Let  $X$  be a smooth well formed Fano weighted complete intersection of multidegree  $(d_1, \dots, d_c)$  in  $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$  of dimension  $n - c \geq 3$ . Important invariants of  $X$  are its anticanonical degree  $(-K_X)^{\dim(X)}$ , the dimension  $h^0(-K_X)$ , and the index  $I(X)$ , which is defined as the maximal number  $i$  such that  $K_X$  is divisible by  $i$  in  $\text{Pic}(X)$ . Since  $\dim X \geq 3$ , the class of the line bundle  $\mathcal{O}_{\mathbb{P}}(1)|_X$  is not divisible in  $\text{Pic}(X)$ , see [Ok16b, Remark 4.2]. Therefore, by Lemma 2.10 one has

$$I(X) = \sum a_i - \sum d_j.$$

For the anticanonical degree of  $X$  one has

$$(-K_X)^{\dim(X)} = \frac{\prod d_j}{\prod a_i} \cdot I(X)^{\dim(X)}.$$

The number  $h^0(-K_X)$  can be computed by Corollary 4.6.

We will use the abbreviation

$$(a_0^{k_0}, \dots, a_m^{k_m}) = (\underbrace{a_0, \dots, a_0}_{k_0 \text{ times}}, \dots, \underbrace{a_m, \dots, a_m}_{k_m \text{ times}}),$$

where  $k_0, \dots, k_m$  will be allowed to be any positive integers. If is some of  $k_i$  is equal to 1 we drop it for simplicity.

Table 1 contains a list of all smooth well formed Fano weighted complete intersections of dimension 4 that are not intersections with linear cones. This list was obtained in [Kü97, Proposition 2.2.1], cf. [BK16, §1.3].

No.	$I$	$\mathbb{P}$	Degrees	$K^4$	$h^0(-K)$	$h^{1,3}$	$h^{2,2}$
1	1	$\mathbb{P}(1^3, 2^2, 3^2)$	6,6	1	3	107	503
2	1	$\mathbb{P}(1^4, 2, 5)$	10	1	4	412	1801
3	1	$\mathbb{P}(1^4, 2^2, 3)$	4,6	2	4	121	572
4	1	$\mathbb{P}(1^5, 4)$	8	2	5	325	1452
5	1	$\mathbb{P}(1^5, 2)$	6	3	5	156	731
6	1	$\mathbb{P}(1^5, 2^2)$	4,4	4	5	75	378
7	1	$\mathbb{P}(1^6, 3)$	2,6	4	6	196	912
8	1	$\mathbb{P}^5$	5	5	6	120	581
9	1	$\mathbb{P}(1^6, 2)$	3,4	6	6	71	364
10	1	$\mathbb{P}^6$	2,4	8	7	77	394
11	1	$\mathbb{P}^6$	3,3	9	7	49	267
12	1	$\mathbb{P}^7$	2,2,3	12	8	42	236
13	1	$\mathbb{P}^8$	2,2,2,2	16	9	27	166
14	2	$\mathbb{P}(1^5, 3)$	6	32	15	70	382
15	2	$\mathbb{P}^5$	4	64	21	21	142
16	2	$\mathbb{P}^6$	2,3	96	27	8	70
17	2	$\mathbb{P}^7$	2,2,2	128	33	3	38
18	3	$\mathbb{P}(1^4, 2, 3)$	6	81	25	24	161
19	3	$\mathbb{P}(1^5, 2)$	4	162	40	5	52
20	3	$\mathbb{P}^5$	3	243	55	1	21
21	3	$\mathbb{P}^6$	2,2	324	70	0	8
22	4	$\mathbb{P}^5$	2	512	105	0	2

Table 1: Fourfold Fano weighted complete intersections.

*Remark 5.1.* Note that there is a misprint in the first line of the table on [Kü97, p. 50]: the varieties described there should be understood as complete intersections of *two* hypersurfaces of weighted degree 6 in  $\mathbb{P}(1^3, 2^2, 3^2)$ . This corresponds to family No. 1 in Table 1.

*Remark 5.2.* The numerical data listed in Table 1 does not describe *every* variety in the corresponding deformation family. For example, a quartic in  $\mathbb{P}^5$  can be seen as a complete intersection of bidegree  $(2, 4)$  in  $\mathbb{P}(1^6, 2)$ , that is an intersection with a linear cone. A non-general variety of the latter type can be contained in a hypersurface of weighted degree 2 whose equation does not depend on the variable of weight 2; such complete intersections cannot be embedded as quartics in  $\mathbb{P}^5$ .

Looking at the anticanonical degrees and dimensions of anticanonical linear systems of varieties from Table 1, we see that varieties from different families are never isomorphic to each other. Similarly, none of them is isomorphic to any of the smooth Fano fourfolds that are zero loci of sections of homogeneous vector bundles on Grassmannians, see [Kü97, Theorem 4.2.1].

*Remark 5.3.* By [Pu98] a general variety from family No. 8 in Table 1 is non-rational. By [To16] a very general variety from families No. 8 and No. 15 is not stably rational. By [HPT16] this is also the case for a very general variety from family No. 19. By [Ok16a, Theorem 1.1] the same holds for very general varieties from families No. 4 and No. 14, by [Ok16a, Corollary 1.4] this holds for very general varieties from families No. 2 and No. 5, and by [Ok16a, Theorem 1.3] this holds for very general varieties from family No. 18. Every variety from families No. 21 and No. 22 is rational for obvious reasons. There are also special rational varieties in families No. 17 and No. 20, but rationality questions for general varieties from these families are far from being settled.

Table 2 contains a list of all smooth well formed Fano weighted complete intersections of dimension 5 that are not intersections with linear cones.

No.	$I$	$\mathbb{P}$	Degrees	$-K^5$	$h^0(-K)$	$h^{1,4}$	$h^{2,3}$
1	1	$\mathbb{P}(1^5, 2, 3, 3)$	6, 6	2	5	354	4594
2	1	$\mathbb{P}(1^6, 5)$	10	2	6	1996	24576
3	1	$\mathbb{P}(1^6, 2, 3)$	4, 6	4	6	359	4758
4	1	$\mathbb{P}(1^7, 4)$	2, 8	4	7	1386	15771
5	1	$\mathbb{P}^6$	6	6	7	455	6055
6	1	$\mathbb{P}(1^7, 2)$	4, 4	8	7	168	2383
7	1	$\mathbb{P}(1^8, 3)$	2, 2, 6	8	8	568	7571
8	1	$\mathbb{P}^7$	2, 5	10	8	294	4074
9	1	$\mathbb{P}^7$	3, 4	12	8	147	2142
10	1	$\mathbb{P}^8$	2, 2, 4	16	9	156	2295
11	1	$\mathbb{P}^8$	2, 3, 3	18	9	88	1364
12	1	$\mathbb{P}^9$	2, 2, 2, 3	24	10	72	1155
13	1	$\mathbb{P}^{10}$	2, 2, 2, 2, 2	32	11	44	759

No.	$I$	$\mathbb{P}$	Degrees	$-K^5$	$h^0(-K)$	$h^{1,4}$	$h^{2,3}$
14	2	$\mathbb{P}(1^4, 2, 2, 3, 3)$	6, 6	32	12	122	1920
15	2	$\mathbb{P}(1^5, 2, 5)$	10	32	16	790	11020
16	2	$\mathbb{P}(1^5, 2, 2, 3)$	4, 6	64	17	117	1936
17	2	$\mathbb{P}(1^6, 4)$	8	64	21	462	6891
18	2	$\mathbb{P}(1^6, 2)$	6	96	22	147	2457
19	2	$\mathbb{P}(1^6, 2, 2)$	4, 4	128	23	44	867
20	2	$\mathbb{P}(1^7, 3)$	2, 6	128	27	183	3072
21	2	$\mathbb{P}^6$	5	160	28	84	1554
22	2	$\mathbb{P}(1^7, 2)$	3, 4	192	29	35	742
23	2	$\mathbb{P}^7$	2, 4	256	35	36	783
24	2	$\mathbb{P}^7$	3, 3	288	36	16	410
25	2	$\mathbb{P}^8$	2, 2, 3	384	43	11	316
26	2	$\mathbb{P}^9$	2, 2, 2, 2	512	51	4	159
27	3	$\mathbb{P}(1^6, 3)$	6	486	57	56	1246
28	3	$\mathbb{P}^6$	4	972	84	7	266
29	3	$\mathbb{P}^7$	2, 3	1458	111	1	83
30	3	$\mathbb{P}^8$	2, 2, 2	1944	138	0	27
31	4	$\mathbb{P}(1^5, 2, 3)$	6	1024	91	16	505
32	4	$\mathbb{P}(1^6, 2)$	4	2048	147	1	90
33	4	$\mathbb{P}^6$	3	3072	203	0	21
34	4	$\mathbb{P}^7$	2, 2	4096	259	0	3
35	5	$\mathbb{P}^6$	2	6250	378	0	0

Table 2: Fivefold Fano weighted complete intersections.

Similarly to Remark 5.2, the numerical data listed in Table 2 does not describe every variety in the corresponding deformation family, but only a general one.

Looking at the anticanonical degrees and dimensions of anticanonical linear systems of varieties from Table 2, we see that varieties from different families are never isomorphic to each other.

*Remark 5.4.* By [Pu98] a general variety from family No. 5 in Table 2 is non-rational. By [To16] a very general variety from this family is not stably rational. By [Pu04] there exist non-rational varieties in family No. 9. By [Ok16a, Theorem 1.1] very general varieties from families No. 2, No. 17, and No. 27 are not stably rational. By [Ok16a, Theorem 1.3] very general varieties from families No. 15 and No. 18 are not stably rational. Every

variety from families No. 34 and No. 35 is rational for obvious reasons. By [Tyu75, Corollary 5.1] every variety from family No. 30 is rational as well.

It would be interesting to study birational geometry of weighted complete intersections from Tables 1 and 2 that are not covered by Remarks 5.3 and 5.4. Also, it would be interesting to study automorphism groups of Fano varieties from Tables 1 and 2, cf. [PrSh16, § A.2]. In particular, it would be interesting to find weighted Fano complete intersections acted on by relatively large automorphism groups, cf. [PS16].

Using the list of index 1 Fano fivefolds provided in Table 2 together with the same trick as in §3, one can compile a list of smooth well formed Calabi–Yau weighted complete intersections of dimension 4 that are not intersections with linear cones. Namely, if a weighted complete intersection of multidegree  $d_1, \dots, d_c$  in  $\mathbb{P}(a_0, \dots, a_n)$  is Calabi–Yau, then a general complete intersection of multidegree  $d_1, \dots, d_c$  in  $\mathbb{P}(1, a_0, \dots, a_n)$  is Fano. For partial classification results concerning Calabi–Yau threefolds see [Og91], [AESZ05], [IMOU16], [IIM16], and references therein.

## APPENDIX A. OPTIMIZATION

The purpose of this section is to prove some bounds on the values of linear functions on special subsets of  $\mathbb{R}^m$ .

Let  $L$  be a real-valued function on a set  $\Omega$ . We say that  $L$  *attains its maximum in  $\Omega$*  if  $L$  is bounded and there is a point  $P$  in  $\Omega$  such that  $L(P) = \sup_{P' \in \Omega} L(P')$ ; in this case we also say that  $L$  attains its maximum in  $\Omega$  at  $P$ .

**Lemma A.1.** *Let  $N$  and  $c$  be positive integers such that  $N > c$ , and  $\alpha$  be a real number. Let  $\tilde{\Omega}$  be a subset of  $\mathbb{R}^{N+c}$  with coordinates  $A_1, \dots, A_N, D_1, \dots, D_c$  defined by inequalities*

$$\begin{aligned} A_1 \cdot \dots \cdot A_N &\leq D_1 \cdot \dots \cdot D_c, \\ A_1 &\geq \dots \geq A_N \geq 1, \quad D_1 \geq \dots \geq D_c, \\ D_1 &\geq 2A_1, \quad D_2 \geq A_2, \quad \dots, \quad D_c \geq A_c. \end{aligned}$$

*Put*

$$L(A_1, \dots, A_N, D_1, \dots, D_c) = \sum_{i=1}^N A_i - \sum_{j=1}^c D_j + \alpha.$$

*Let  $\Omega \subset \tilde{\Omega}$  be a non-empty closed subset. Then the function  $L$  attains its maximum in  $\Omega$ .*

*Proof.* It is enough to prove the assertion for  $\alpha = 0$ .

Let  $M$  be a real number, and let

$$\Omega' \subset \Omega \cap \{(A_1, \dots, D_c) \mid A_1 \leq M\}$$

be a non-empty closed subset. We claim that  $L$  attains its maximum in  $\Omega'$ . Indeed, let  $\bar{P}$  be some point of  $\Omega$ , and put  $\bar{L} = L(\bar{P})$ . If  $D_1 > NM - \bar{L}$ , then

$$L(A_1, \dots, D_c) = \sum_{i=1}^N A_i - \sum_{j=1}^c D_j \leq NA_1 - D_1 < N(A_1 - M) + \bar{L} \leq \bar{L}.$$

Thus  $L$  attains its maximum in  $\Omega'$  if and only if it attains its maximum in the closed subset

$$\Omega'' = \Omega' \cap \{(A_1, \dots, D_c) \mid D_1 \leq NM - \bar{L}\}$$

containing  $\bar{P}$ . It remains to notice that  $\Omega''$  is a compact subset of  $\mathbb{R}^{N+c}$ .

We conclude from the above argument that  $L$  attains its maximum in  $\Omega$  if and only if it attains its maximum in the closed subset

$$\Omega^{(1)} = \Omega \cap \{(A_1, \dots, D_c) \mid A_1 \geq -L_0\}.$$

Let  $P_0$  be some point of  $\Omega$ , and put  $L_0 = L(P_0)$ . If  $A_{c+1} < (A_1 + L_0) \cdot (N - c)^{-1}$ , then

$$\begin{aligned} L(A_1, \dots, D_c) &= \sum_{i=1}^N A_i - \sum_{j=1}^c D_j = (A_1 - D_1) + \sum_{j=2}^c (A_j - D_j) + \sum_{i=c+1}^N A_i \leq \\ &\leq -A_1 + (N - c)A_{c+1} < -A_1 + A_1 + L_0 = L_0. \end{aligned}$$

Thus  $L$  attains its maximum in  $\Omega^{(1)}$  if and only if it attains its maximum in the closed subset

$$\Omega^{(2)} = \Omega^{(1)} \cap \left\{ (A_1, \dots, D_c) \mid A_{c+1} \geq \frac{A_1 + L_0}{N - c} \right\}$$

containing  $P_0$ .

Suppose that  $(A_1, \dots, D_c) \in \Omega^{(2)}$ . Then

$$D_1^c \geq D_1 \cdot \dots \cdot D_c \geq A_1 \cdot \dots \cdot A_N \geq A_1 \cdot \dots \cdot A_{c+1} \geq \left( \frac{A_1 + L_0}{N - c} \right)^{c+1},$$

so that

$$D_1 \geq \left( \frac{1}{N - c} \right)^{\frac{c+1}{c}} \cdot (A_1 + L_0)^{\frac{c+1}{c}}.$$

One has

$$\begin{aligned} \text{(A.1)} \quad L(A_1, \dots, D_c) &= \sum_{i=1}^N A_i - \sum_{j=1}^c D_j \leq N \cdot A_1 - D_1 = N(A_1 + L_0) - D_1 - NL_0 \leq \\ &\leq (A_1 + L_0) \cdot \left( N - \left( \frac{1}{N - c} \right)^{\frac{c+1}{c}} \cdot (A_1 + L_0)^{\frac{1}{c}} \right) - NL_0. \end{aligned}$$

Put

$$M = \max \{ |(N + 1)L_0| - L_0, (N + 1)^c \cdot (N - c)^{c+1} - L_0 \}.$$

If  $A_1 > M$ , then the right hand side of (A.1) is less than  $L_0$ . Thus  $L$  attains its maximum in  $\Omega^{(2)}$  if and only if it attains its maximum in the closed subset

$$\Omega^{(3)} = \Omega^{(2)} \cap \{(A_1, \dots, D_c) \mid A_1 \leq M\},$$

and we are done by the argument from the beginning of the proof.  $\square$

*Remark A.2.* One can easily modify the assertion of Lemma A.1 so that the condition  $N > c$  will not be important.

The following theorem will be our main technical tool to find the points where certain functions attain their maximal values. It is well known as the *method of Lagrange multipliers*, or sometimes the *Kuhn–Tusker Theorem*.



**Theorem A.3** (see [Va92, p. 503] or [MWG95, Theorem M.K.2]). *Let  $G_1, \dots, G_p$  be differentiable functions on  $\mathbb{R}^m$  with coordinates  $x_1, \dots, x_m$ . Let  $\Omega \subset \mathbb{R}^m$  be a subset defined by inequalities  $G_i \leq 0$ ,  $1 \leq i \leq p$ . Let  $L$  be a differentiable function on  $\mathbb{R}^m$ . Suppose that  $L$  attains its maximum in  $\Omega$  at a point  $P$ . Then*

$$\left( \frac{\partial L}{\partial x_1}, \dots, \frac{\partial L}{\partial x_m} \right) [P] = \sum_{i=1}^p \lambda_i \left( \frac{\partial G_i}{\partial x_1}, \dots, \frac{\partial G_i}{\partial x_m} \right) [P]$$

for some non-negative numbers  $\lambda_i$ . Moreover, if for some  $j$  one has  $\lambda_j \neq 0$ , then  $G_j(P) = 0$ .

To proceed we need to establish some elementary inequalities that will be used in the proof of Proposition A.11.

**Lemma A.4.** *Let  $l \geq 1$  be an integer, and  $r$  be a non-negative integer. Let  $\Omega \subset \mathbb{R}_{B,D}^2$  be defined by inequalities  $B^l \leq 2D^r$  and  $D \geq B \geq 1$ . Put*

$$L(B, D) = lB - rD - (l + 1).$$

*Then  $L$  is non-positive on  $\Omega$ . Moreover,  $L(B, D)$  is negative unless  $l = 1$ ,  $r = 0$ , and  $B = 2$ .*

*Proof.* Suppose that  $l = 1$ . If  $r > 0$ , then

$$L(B, D) = B - rD - 2 \leq B - D - 2 \leq -2 < 0.$$

If  $r = 0$ , then  $B \leq 2$ , and

$$L(B, D) = B - 2 \leq 0.$$

Moreover, in this case  $L(B, D) = 0$  if and only if  $B = 2$ . Thus we will assume that  $l \geq 2$ .

The function  $L$  attains its maximum in  $\Omega$  at some point  $P$ . To see this apply Lemma A.1 with  $c = r + 1$ , an arbitrary  $N \geq \max\{l + 1, c + 1\}$ , and the closed subset defined by conditions

$$A_1 = \dots = A_{l+1}, \quad A_{l+2} = \dots = A_N = 1, \quad D_1 = 2A_1, \quad D_2 = \dots = D_c.$$

Abusing notation a little bit, we write  $P = (B, D)$  and put  $M = L(P)$ .

If  $B = 1$ , then

$$L(B, D) = l - rD - (l + 1) = -rD - 1 < 0.$$

Thus we will assume that  $B > 1$ .

If  $l \leq r$ , then

$$M = lB - rD - (l + 1) \leq (l - r)B - (l + 1) < 0.$$

Thus we will assume that  $l > r$ .

Suppose that  $r = 0$ . Then  $B^l \leq 2$ . Since  $l > 1$ , this implies that  $B < 1 + \frac{1}{l}$ . This gives

$$M = lB - rD - (l + 1) < l \cdot \left( 1 + \frac{1}{l} \right) - (l + 1) = 0.$$

Thus we will assume that  $r > 0$ .

Similarly, suppose that  $B = D$ . Then  $B^{l-r} \leq 2$ , which implies  $B \leq 1 + \frac{1}{l-r}$ . This gives

$$M = lB - rD - (l + 1) = (l - r)B - (l + 1) \leq (l - r) \cdot \left( 1 + \frac{1}{l - r} \right) - (l + 1) = -r < 0.$$

Thus we will assume that  $D > B$ .

By Theorem A.3 one has

$$(A.2) \quad (l, -r) = \lambda(lB^{l-1}, -2rD^{r-1})$$

for some non-negative number  $\lambda$ . Actually, (A.2) implies that

$$\lambda = \frac{1}{B^{l-1}},$$

so that  $\lambda$  is positive and  $B^l = 2D^r$  by Theorem A.3. Also, since  $r > 0$ , equation (A.2) implies that

$$1 = 2D^{r-1} \cdot \lambda = \frac{2D^{r-1}}{B^{l-1}} = \frac{B}{D} < 1,$$

which is a contradiction.  $\square$

**Lemma A.5.** *Let  $k$  be a positive integer,  $l$  be a non-negative integer, and  $p$  be a non-negative real number such that  $k+l+p > 2$ ; let  $r$  be a non-negative integer. Let  $\Omega \subset \mathbb{R}_{A,B,D}^3$  be defined by inequalities  $A^{k-1}B^l \leq 2D^r$ ,  $A \geq B \geq 1$ ,  $A \geq k+l+p$ ,  $D \geq B$ , and an additional inequality  $D \geq A$  in the case when  $k \geq 2$ . Put*

$$L(A, B, D) = (k-2)A + lB - rD + p.$$

*Then  $L$  is non-positive on  $\Omega$ . Moreover,  $L(A, B, D)$  is negative unless  $A = p+2$  and  $p > 0$ .*

*Proof.* The function  $L$  attains its maximum in  $\Omega$  at some point  $P$ . To see this apply Lemma A.1 with  $c = r+1$ , an arbitrary  $N \geq \max\{k+l, c+1\}$ , and the closed subset defined by conditions

$$\begin{aligned} A_1 = \dots = A_k, \quad A_{k+1} = \dots = A_{k+l}, \quad A_{k+l+1} = \dots = A_N = 1, \\ D_1 = 2A_1, \quad D_2 = \dots = D_c, \quad A_1 \geq k+l+p. \end{aligned}$$

Abusing notation, we write  $P = (A, B, D)$  and put  $M = L(P)$ .

If  $k = 1$ , we have  $B^l \leq 2D^r$ , so that

$$M = lB - rD + (p - A) \leq lB - rD - (l+1) \leq 0$$

by Lemma A.4. Moreover, if  $M = 0$ , then  $l = 1$ ,  $r = 0$ , and  $B = 2$ , so  $M = p+2-A$  and  $A = p+2$ . In particular, condition  $k+l+p > 2$  implies that  $p > 0$ . Thus we will assume that  $k \geq 2$ .

Note that  $r > 0$ . Indeed, otherwise one has

$$2 \geq A^{k-1}B^l \geq A^{k-1} \geq A > 2,$$

which is absurd.

If  $l = 0$ , then  $A^{k-1} \leq 2D^r$ , so that

$$M = (k-2)A - rD + p = ((k-1)A - rD - k) + (k+p-A) < 0$$

by Lemma A.4. Thus we will assume that  $l > 0$ .

If  $B = 1$  then  $A^{k-1} \leq 2D^r$ , so that

$$M = (k-1)A + l - rD - A + p = ((k-1)A - rD - k) + (k+l+p-A) < 0$$

by Lemma A.4. Thus we will assume that  $B > 1$ .

If  $A = B$  then  $A^{k+l-1} \leq 2D^r$ , so that

$$M = (k+l-2)A - rD + p = ((k+l-1)A - rD - (k+l)) + (k+l+p-A) < 0$$

by Lemma A.4. Thus we will assume that  $A > B$ . In particular, we have  $D > B$ .

Suppose that  $D = A$ . If  $r < k - 1$ , then  $A^{k-1-r}B^l \leq 2$ , so that  $A \leq 2 < k + l + p$ , a contradiction. Hence  $r \geq k - 1$ , and  $B^l \leq 2D^{r-k+1}$ . We have

$$M = lB + (k - r - 2)D + p = (lB - (r - k + 1)D - (l + 1)) + (l + p + 1 - D) < 0$$

by Lemma A.4. Thus we will assume that  $D > A$ .

By Theorem A.3 one has

$$(A.3) \quad (k - 2, l, -r) = \lambda_1((k - 1)A^{k-2}B^l, lA^{k-1}B^{l-1}, -2rD^{r-1}) + \lambda_2(-1, 0, 0)$$

for some non-negative numbers  $\lambda_1$  and  $\lambda_2$ . Since  $l > 0$ , equation (A.3) implies that

$$\lambda_1 = \frac{1}{A^{k-1}B^{l-1}},$$

so that  $\lambda_1$  is positive and  $A^{k-1}B^l = 2D^r$  by Theorem A.3. Finally, since  $r > 0$ , equation (A.3) implies that

$$1 = 2D^{r-1} \cdot \lambda_1 = \frac{2D^{r-1}}{A^{k-1}B^{l-1}} = \frac{B}{D} < 1,$$

which is a contradiction.  $\square$

**Lemma A.6.** *Let  $k$  and  $c$  be positive integers, and  $l$  be a non-negative real number such that  $k + l > 2$ . Let  $\Omega$  be a subset of  $\mathbb{R}^2$  with coordinates  $A, D$  defined by inequalities  $A^k \leq D^c$ ,  $D \geq 2A$ , and  $A \geq k + l$ . Put*

$$L(A, D) = kA - cD + l.$$

*Then  $L$  is negative on  $\Omega$ .*

*Proof.* The function  $L$  attains its maximum in  $\Omega$  at some point  $P$ . To see this apply Lemma A.1 with an arbitrary  $N \geq \max\{k, c + 1\}$ , and the closed subset defined by conditions

$$A_1 = \dots = A_k, \quad A_{k+1} = \dots = A_N = 1, \quad D_1 = \dots = D_c, \quad A_1 \geq k + l.$$

Abusing notation, we write  $P = (A, D)$ . Note that if  $(k + 1)A \leq cD$ , then

$$L(A, D) = kA - cD + l \leq l - A < 0.$$

In particular, this happens if  $k < 2c$ , since  $k$  and  $c$  are integers.

By Theorem A.3 one has

$$(k, -c) = \lambda_1(kA^{k-1}, -cD^{c-1}) + \lambda_2(2, -1) + \lambda_3(-1, 0)$$

for some non-negative numbers  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$ .

Suppose that  $\lambda_1 > 0$ . By Theorem A.3 this means that  $A^k = D^c$ . We can assume that  $k \geq 2c$ . Thus  $D^c \geq A^{2c}$ , and  $D \geq A^2$ .

Assume that  $c \geq 2$ . Then  $k + 1 \leq ck \leq c(k + l) \leq cA$ , so that

$$(k + 1)A \leq cA^2 \leq cD,$$

and we are done. Hence we have  $c = 1$ , and

$$L(A, D) = kA - D + l \leq kA - A^2 + l \leq l - lA < 0.$$

Therefore, we may suppose that  $\lambda_1 = 0$ . Then

$$(A.4) \quad (k, -c) = (2\lambda_2 - \lambda_3, -\lambda_2),$$

so that  $\lambda_2 = c$  and  $D = 2A$  by Theorem A.3. Also, we see from (A.4) that  $k \leq 2c$ . If  $k = 2c$ , then  $A^{2c} = (2A)^c$ , which means  $A = 2 < k + l$ , a contradiction. Thus, we have  $k < 2c$ , which implies the assertion of the lemma.  $\square$

**Lemma A.7.** *Let  $k$  be a positive integer, let  $l$  be a non-negative integer such that  $k+l > 2$ , and let  $c$  be a non-negative integer. Let  $\Omega \subset \mathbb{R}_{A,B,D}^3$  be defined by inequalities  $A^k B^l \leq D^c$ ,  $D \geq 2A$ ,  $A \geq B \geq 1$ , and  $A \geq k+l$ . Put*

$$L(A, B, D) = kA + lB - cD.$$

*Then  $L$  is negative on  $\Omega$ .*

*Proof.* The function  $L$  attains its maximum in  $\Omega$  at some point  $P$ . To see this apply Lemma A.1 with an arbitrary  $N \geq \max\{k+l, c+1\}$ , and the closed subset defined by conditions

$$A_1 = \dots = A_k, \quad A_{k+1} = \dots = A_{k+l}, \quad A_{k+l+1} = \dots = A_N = 1, \\ D_1 = \dots = D_c, \quad A_1 \geq k+l.$$

Abusing notation, we write  $P = (A, B, D)$  and put  $M = L(P)$ .

Note that  $c > 0$ . Indeed, otherwise one has

$$1 \geq A^k B^l \geq A^k \geq A > 2,$$

which is absurd.

If  $l = 0$ , then  $A^k \leq D^c$ , so that

$$M = kA - cD < 0$$

by Lemma A.6. Thus we will assume that  $l > 0$ .

If  $A = B$ , then  $A^{k+l} \leq D^c$ , so that

$$M = (k+l)A - cD < 0$$

by Lemma A.6. Thus we will assume that  $A > B$ .

If  $B = 1$ , then  $A^k \leq D^c$ , so that

$$M = kA - cD + l < 0$$

by Lemma A.6. Thus we will assume that  $B > 1$ .

If  $D = 2A$ , then

$$M = kA + lB - c \cdot 2A = (k-2)A + lB - (c-1) \cdot 2A < 0$$

by Lemma A.5. Thus we will assume that  $D > 2A$ .

By Theorem A.3 one has

$$(A.5) \quad (k, l, -c) = \lambda_1(kA^{k-1}B^l, lA^k B^{l-1}, -cD^{c-1}) + \lambda_2(-1, 0, 0)$$

for some non-negative numbers  $\lambda_1$  and  $\lambda_2$ . Since  $l > 0$ , equation (A.5) implies that

$$\lambda_1 = \frac{1}{A^k B^{l-1}},$$

so that  $\lambda_1$  is positive and  $A^k B^l = D^c$  by Theorem A.3. Finally, since  $c > 0$ , equation (A.5) implies that

$$1 = D^{c-1} \cdot \lambda_1 = \frac{D^{c-1}}{A^k B^{l-1}} = \frac{B}{D} < 1,$$

which is a contradiction.  $\square$

Consider a real vector space  $\mathbb{R}^m$ . For a vector  $v \in \mathbb{R}^m$  we denote by  $v^{(i)}$  its  $i$ -th coordinate. Denote

$$\mathbf{e}_i = (\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0) \in \mathbb{R}^m, \quad 1 \leq i \leq m.$$

**Lemma A.8.** *Let  $u \in \mathbb{R}^m$  be a vector such that*

$$0 \leq u^{(1)} \leq \dots \leq u^{(m)}.$$

*Put  $u_{-1} = -\mathbf{e}_1$ , put  $u_0 = \mathbf{e}_1$ , put  $u_i = -\mathbf{e}_i + \mathbf{e}_{i+1}$  for  $1 \leq i \leq m-1$ , and put  $u_m = -\mathbf{e}_m$ . Choose a subset  $\Lambda \subset \{-1, 0, \dots, m\}$ . Suppose that*

$$\lambda u + \sum_{i \in \Lambda} \lambda_i u_i = (1, \dots, 1)$$

*for some non-negative number  $\lambda$  and some positive numbers  $\lambda_i$ . Then there exist two indices  $0 \leq p \leq q \leq m$  such that  $u^{(p+1)} = \dots = u^{(q)}$  and*

$$\{0, \dots, p-1\} \cup \{q+1, \dots, m\} \subset \Lambda.$$

*Proof.* Note that for the vector  $\lambda u$  the same assumptions hold as for the vector  $u$  itself provided that  $\lambda \geq 0$ . Thus we will replace  $u$  by  $\lambda u$  and assume that  $\lambda = 1$ . In other words we have a system of equations

$$\begin{cases} u^{(1)} - \lambda_{-1} + \lambda_0 - \lambda_1 = 1, \\ u^{(2)} + \lambda_1 - \lambda_2 = 1, \\ \dots \\ u^{(m-1)} + \lambda_{m-2} - \lambda_{m-1} = 1, \\ u^{(m)} + \lambda_{m-1} - \lambda_m = 1. \end{cases}$$

Choose the indices  $p$  and  $q$  so that

$$u^{(p)} < u^{(p+1)} = \dots = u^{(q)} = 1 < u^{(q+1)}.$$

In particular, we put  $p = q$  if one has  $u^{(p)} < 1 < u^{(p+1)}$ , we put  $p = q = 0$  if  $1 < u^{(1)}$ , and we put  $p = q = m$  if  $u^{(m)} < 1$ .

For  $1 < i \leq p$  we have

$$1 = u^{(i)} + \lambda_{i-1} - \lambda_i < 1 + \lambda_{i-1},$$

so  $\lambda_{i-1} > 0$ . Moreover, if  $p > 0$ , then

$$1 = u^{(1)} + \lambda_{-1} + \lambda_0 - \lambda_1 < 1 + \lambda_0,$$

so  $\lambda_0 > 0$ . In the same way for  $j > q$  we have

$$1 = u^{(j)} + \lambda_{j-1} - \lambda_j > 1 - \lambda_j,$$

so  $\lambda_j > 0$ . This exactly gives the assertion of the lemma.  $\square$

**Lemma A.9.** *Let  $u \in \mathbb{R}^m$  be a vector such that*

$$0 \geq u^{(1)} \geq \dots \geq u^{(m)}.$$

Put  $u_0 = -\mathbf{e}_1$ , and put  $u_i = -\mathbf{e}_i + \mathbf{e}_{i+1}$  for  $1 \leq i \leq m-1$ . Choose a subset  $\Lambda \subset \{0, \dots, m-1\}$ . Suppose that

$$\lambda u + \sum_{i \in \Lambda} \lambda_i u_i = (-1, \dots, -1)$$

for some non-negative number  $\lambda$  and some positive numbers  $\lambda_i$ . Then one of the following possibilities occurs:

- (i)  $0 \in \Lambda$  and  $u^{(2)} = \dots = u^{(m)}$ ;
- (ii)  $u^{(1)} = \dots = u^{(m)}$ ;
- (iii)  $\{1, 2, \dots, m-1\} \subset \Lambda$ ;
- (iv)  $\{0, 2, 3, \dots, m-1\} \subset \Lambda$ .

*Proof.* Note that for the vector  $\lambda u$  the same assumptions hold as for the vector  $u$  itself provided that  $\lambda \geq 0$ . Thus we will replace  $u$  by  $\lambda u$  and assume that  $\lambda = 1$ . In other words we have a system of equations

$$\begin{cases} u^{(1)} - \lambda_0 - \lambda_1 = -1, \\ u^{(2)} + \lambda_1 - \lambda_2 = -1, \\ \dots \\ u^{(m-1)} + \lambda_{m-2} - \lambda_{m-1} = -1, \\ u^{(m)} + \lambda_{m-1} = -1. \end{cases}$$

Suppose that  $u^{(m)} = -1$ . Let  $q$  be the minimal index such that  $u^{(q)} = -1$ . Then, considering equations from the last one to the  $q$ -th one by one we have

$$\lambda_{m-1} = \dots = \lambda_{q-1} = 0.$$

Moreover, if  $q > 2$ , then

$$u^{(q-1)} + \lambda_{q-2} - \lambda_{q-1} = u^{(q-1)} + \lambda_{q-2} > -1,$$

which is impossible. Thus either  $q = 2$ , so  $u^{(1)} > -1$  and  $\lambda_0 > 0$ , which corresponds to case (i), or  $q = 1$ , which corresponds to case (ii).

Now suppose that  $u^{(m)} < -1$ . Choose the indices  $1 \leq p \leq q < m$  such that

$$u^{(p)} > u^{(p+1)} = \dots = u^{(q)} = -1 > u^{(q+1)}.$$

Then for  $i > q$  one has

$$-1 = u^{(i)} + \lambda_{i-1} - \lambda_i < -1 + \lambda_{i-1},$$

so  $\lambda_{i-1} > 0$ . Moreover, for  $p \leq j \leq q$  one has

$$-1 = u^{(j)} + \lambda_{j-1} - \lambda_j = -1 + \lambda_{j-1} - \lambda_j < -1 + \lambda_{j-1},$$

so we also have  $\lambda_{j-1} > 0$ .

If one has  $\lambda_i > 0$  for all  $1 \leq i \leq p-1$ , then we obtain case (iii). Otherwise take the maximal number  $s$  with  $1 \leq s \leq p-1$  such that  $\lambda_s = 0$ . If  $s > 1$  then

$$u^{(s)} + \lambda_{s-1} - \lambda_s > -1,$$

which is impossible. Thus  $s = 1$  and  $\lambda_0 > 0$ , which gives us case (iv).  $\square$

**Lemma A.10.** Choose a vector  $u = (u^{(1)}, \dots, u^{(N+c)}) \in \mathbb{R}^{N+c}$  such that

$$0 \leq u^{(1)} \leq \dots \leq u^{(N)}, \quad 0 \geq u^{(N+1)} \geq \dots \geq u^{(N+c)}.$$

Put  $u_{-1} = -\mathbf{e}_1$  and  $u_0 = 2\mathbf{e}_1 - \mathbf{e}_{N+1}$ . Furthermore, put  $u_i = -\mathbf{e}_i + \mathbf{e}_{i+1}$  for  $1 \leq i \leq N-1$  and for  $N+1 \leq i < N+c$ . Finally, put  $u_N = -\mathbf{e}_N$ . Choose a subset  $\Lambda' \subset \{-1, 0, \dots, N+c-1\}$ . Suppose that

$$\lambda u + \sum_{i \in \Lambda'} \lambda_i u_i = (\underbrace{1, \dots, 1}_N, \underbrace{-1, \dots, -1}_c)$$

for some non-negative number  $\lambda$  and some positive numbers  $\lambda_i$ . Define

$$\Lambda'' = \{i \mid u^{(i)} = u^{(i+1)}\}.$$

Then one of the following possibilities occurs:

(I) there is an index  $0 \leq q \leq N$  such that

$$\{1, \dots, q-1, q+1, \dots, N, N+1, \dots, N+c-1\} \subset \Lambda' \cup \Lambda''.$$

and  $N \in \Lambda'$ ;

(II) there is an index  $1 \leq p \leq N$  such that

$$\{0, \dots, p-1, p+1, \dots, N-1, N+2, \dots, N+c-1\} \subset \Lambda' \cup \Lambda''$$

and  $0 \in \Lambda'$ ;

(III) there are indices  $1 \leq p \leq q \leq N-1$  such that

$$\{0, \dots, p-1, p+1, \dots, q-1, q+1, \dots, N, N+2, \dots, N+c-1\} \subset \Lambda' \cup \Lambda''$$

and  $\{0, N\} \subset \Lambda'$ .

*Proof.* Apply Lemma A.8 to the first  $N$  coordinates of the vector  $u$  and Lemma A.9 to its last  $c$  coordinates. Define numbers  $p$  and  $q$  following the notation of Lemma A.8. The only possibility to have  $0 \notin \Lambda$  is to have  $p = 0$ . Then  $\{1, \dots, q-1\} \subset \Lambda''$  and  $\{q+1, \dots, N\} \subset \Lambda'$ . Moreover, for the last  $c$  coordinates only cases (ii) or (iii) from Lemma A.9 can occur, which gives us case (I).

So we can assume that  $0 \in \Lambda'$ . From Lemma A.9 one can easily see that

$$\{N+2, \dots, N+c-1\} \subset \Lambda' \cup \Lambda''.$$

If  $q = N$  then  $\{0, \dots, p-1\} \subset \Lambda'$  and  $\{p+1, \dots, N-1\} \subset \Lambda''$ , and we obtain case (II).

If  $q < N$  then

$$\{0, \dots, p-1, q+1, \dots, N\} \subset \Lambda'$$

and  $\{p+1, \dots, q-1\} \subset \Lambda''$ , and we obtain case (III). □

Now we are ready to prove the main result of this section.

**Proposition A.11.** Let  $c$  and  $N$  be positive integers such that  $N \geq 2c+1$ . Let  $\Omega$  be a subset of  $\mathbb{R}^{N+c}$  with coordinates  $A_1, \dots, A_N, D_1, \dots, D_c$  defined by inequalities

$$\begin{aligned} A_1 \cdot \dots \cdot A_N &\leq D_1 \cdot \dots \cdot D_c, \\ A_1 &\geq \dots \geq A_N \geq 1, \quad D_1 \geq \dots \geq D_c, \\ D_1 &\geq 2A_1, \quad D_2 \geq A_2, \quad \dots, \quad D_c \geq A_c. \end{aligned}$$

Put

$$L(A_1, \dots, A_N, D_1, \dots, D_c) = A_1 + \dots + A_N - D_1 - \dots - D_c.$$

Then  $L$  is non-positive on the subset of  $\Omega$  where  $A_1 \geq N$ .

*Proof.* Rewrite the inequalities defining  $\Omega$  as

$$\begin{aligned} A_1 \cdot \dots \cdot A_N - D_1 \cdot \dots \cdot D_c &\leq 0, \\ A_2 - A_1 &\leq 0, \dots, A_N - A_{N-1} \leq 0, 1 - A_N \leq 0, \\ D_2 - D_1 &\leq 0, \dots, D_c - D_{c-1} \leq 0, \\ A_2 - D_2 &\leq 0, \dots, A_c - D_c \leq 0, 2A_1 - D_1 \leq 0. \end{aligned}$$

The function  $L$  attains its maximum in  $\Omega$  at some point  $P \in \Omega$  by Lemma A.1. Abusing notation a little bit, we write  $P = (A_1, \dots, A_N, D_1, \dots, D_c)$ . If for some  $2 \leq i \leq c$  one has  $A_i = D_i$ , we cancel  $A_i$  and  $D_i$  from the inequalities defining  $\Omega$  and from the definition of  $L$  and arrive to the same assertion with a smaller number of parameters. Therefore, we assume that for all  $2 \leq i \leq c$  one has  $A_i < D_i$  (in particular, this is the case when  $c = 1$ ). Note that after such cancelation the condition  $N \geq 2c + 1$  is preserved.

Denote  $\Pi_A = A_1 \cdot \dots \cdot A_N$  and  $\Pi_D = D_1 \cdot \dots \cdot D_c$ . Applying Theorem A.3 and keeping in mind that  $A_i < D_i$  for  $2 \leq i \leq c$ , we obtain an equality

$$\begin{aligned} (\underbrace{1, \dots, 1}_N, \underbrace{-1, \dots, -1}_c) &= \lambda \left( \frac{\Pi_A}{A_1}, \dots, \frac{\Pi_A}{A_N}, \frac{\Pi_D}{D_1}, \dots, \frac{\Pi_D}{D_c} \right) + \\ &+ \sum_{i=1}^{N-1} \lambda_i (\underbrace{0, \dots, 0}_{i-1}, -1, 1, 0, \dots, 0) + \lambda_N (\underbrace{0, \dots, 0}_{N-1}, -1, 0, \dots, 0) + \\ &+ \sum_{i=N+1}^{N+c-1} \lambda_i (\underbrace{0, \dots, 0}_{i-1}, -1, 1, 0, \dots, 0) + \lambda_0 (\underbrace{2, 0, \dots, 0}_{N-1}, -1, 0, \dots, 0) + \lambda_{-1} (-1, 0, \dots, 0) \end{aligned}$$

for some non-negative numbers  $\lambda$  and  $\lambda_i$ , where  $-1 \leq i \leq N + c - 1$ .

Let  $\Lambda' \subset \{-1, 0, 1, \dots, N + c - 1\}$  be the set of indices such that for any  $i \in \Lambda'$  one has  $\lambda_i > 0$ , and let  $\Lambda'' = \{i \mid u^{(i)} = u^{(i+1)}\}$ . Keeping in mind that by Theorem A.3 for any  $i \in \Lambda'$  the corresponding inequality turns into equality, we see that by Lemma A.10 we have the following possibilities:

(I)

$$(A_1, \dots, A_N, D_1, \dots, D_c) = (\underbrace{A, \dots, A}_k, \underbrace{B, \dots, B}_l, \underbrace{D, \dots, D}_c)$$

for some  $k \geq 1$  and  $l \geq 0$ ;

(II)

$$(A_1, \dots, A_N, D_1, \dots, D_c) = (\underbrace{A, \dots, A}_k, \underbrace{B, \dots, B}_l, 2A, \underbrace{D, \dots, D}_{c-1})$$

for some  $k \geq 1$  and  $l \geq 0$ ;

(III)

$$(A_1, \dots, A_N, D_1, \dots, D_c) = (\underbrace{A, \dots, A}_k, \underbrace{B, \dots, B}_l, \underbrace{1, \dots, 1}_p, 2A, \underbrace{D, \dots, D}_{c-1})$$



for some  $k \geq 1$ ,  $l \geq 1$ , and  $p \geq 1$ .

In all cases one has  $c \geq 1$  and  $N \geq 2c + 1 \geq 3$ .

In case (I) we have  $L(P) < 0$  provided that  $A_1 \geq N$ ; to see this apply Lemma A.7. In case (II) we have  $L(P) < 0$  provided that  $A_1 \geq N$ ; to see this apply Lemma A.5 with  $r = c - 1$  and  $p = 0$ . Finally, in case (III) we have  $L(P) \leq 0$  provided that  $A_1 \geq N$ ; to see this apply Lemma A.5 with  $r = c - 1$ .  $\square$

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